The Mathematics of Sunflowers

DRAFT

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Many stems, cones, and compound flowers have successive growths such as branches, leaves, petals, and seeds separated by the same angle. This angle could have evolved to ensure that successive leaves overlap as little as possible. This angle must not be a rational fraction of 360° since any rational of the from n/m would result in complete overlap after m leaves or fewer. In some sense, the golden ratio is "best" and is often found in nature. That is, the angle between outgrowths is $2\pi\varphi$ radians where φ is the golden ratio:

$\varphi = (1+\sqrt{5})/2 = 1.618033989...$

$2\pi\varphi$ radians is equivalent to 10.16640738... radians, or in degrees 582.492236...°

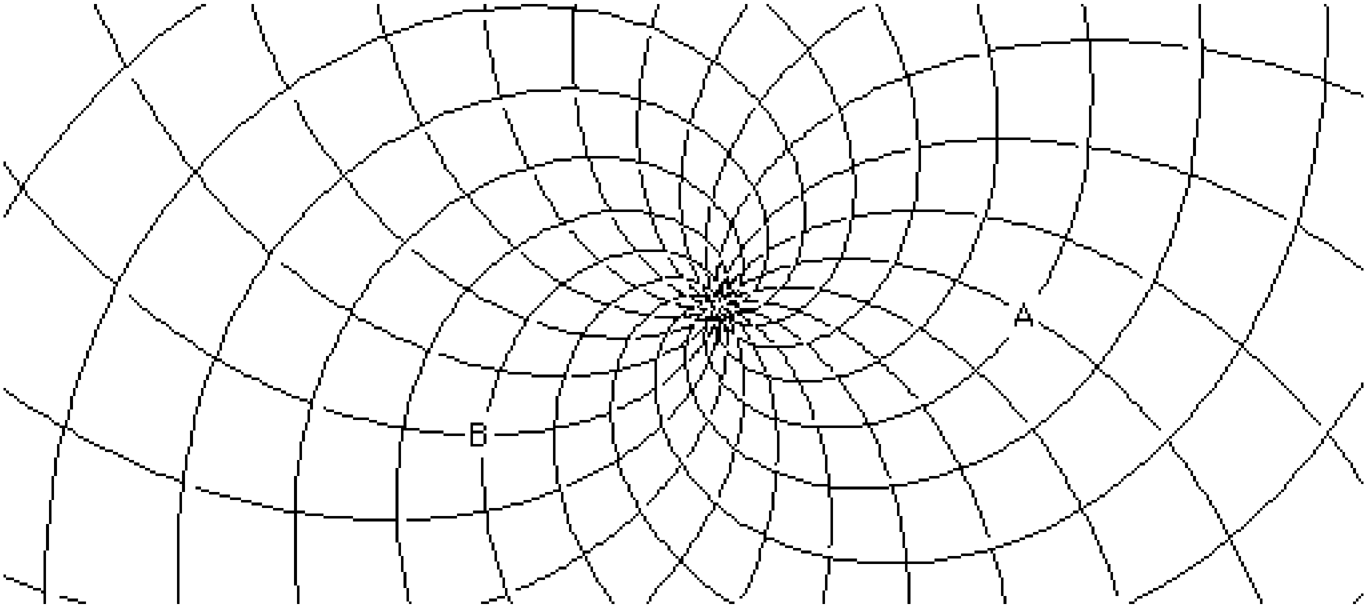
Actually, angles are measured modulus 2π so $2\pi\varphi$ is equivalent to $2\pi(\varphi - 1)$. One of the ways of defining φ is through

$1/\varphi = \varphi - 1 = .618033989..$

Thus, the angle between leaves is also $2\pi/\varphi = 3.883222077$ radians or 222.492236...°

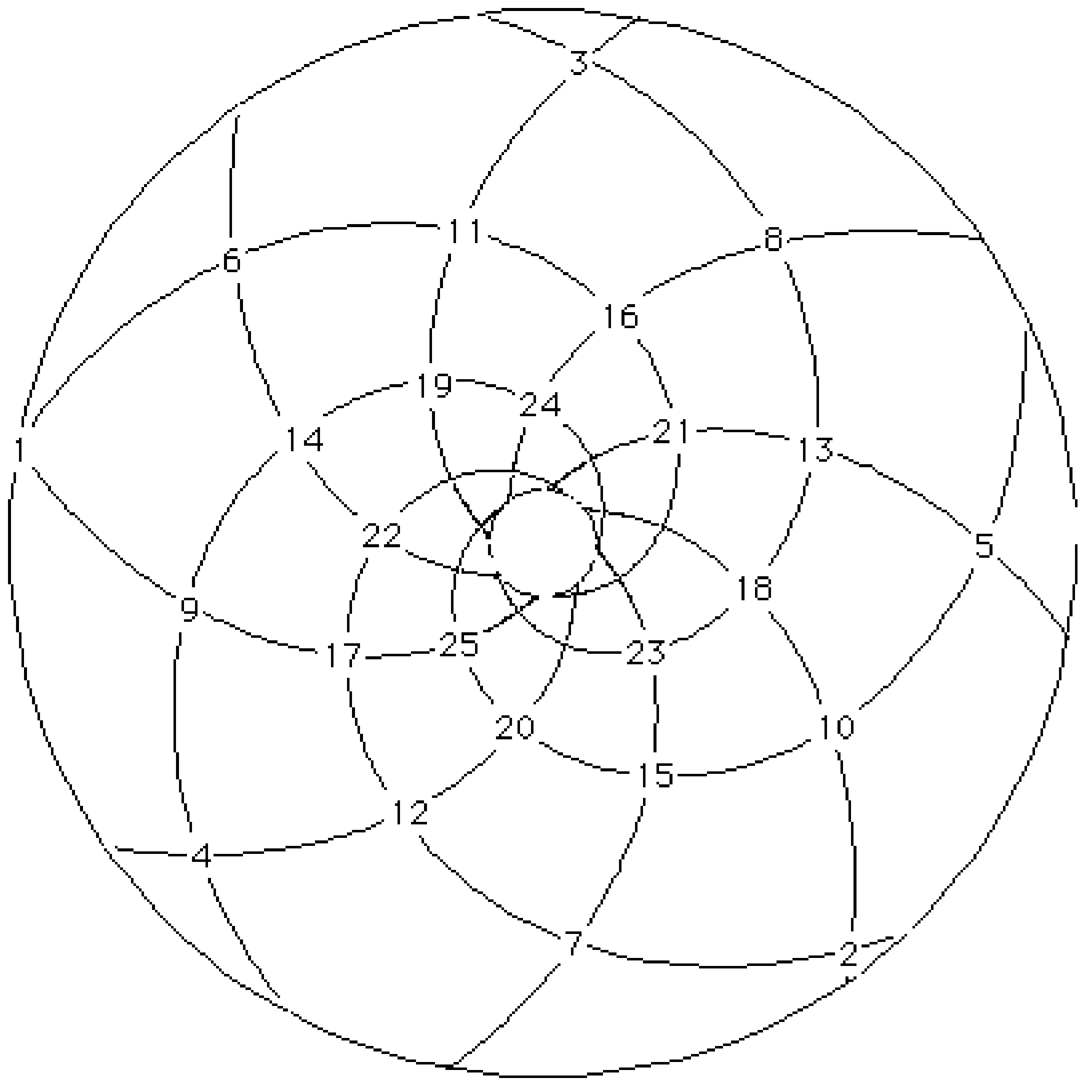
Finally, on a plant you could measure angles in either direction, so one would naturally report an angle less than 180° or π radians. So, while the plant might have actually spun $2\pi/\varphi$ radians you would measure $2\pi - 2\pi/\varphi = 2\pi(1 - 1/\varphi) = 2\pi/\varphi^2$ radians or 137.507764...°. This is known as the **golden angle**. I am told that this angle is often approximately observed in nature! For instance, in the sunflower, the seeds, if measured in order of their distance from the center, are separated by this angle.

If you look at the seeds in a sunflower, you see what appears to be a pattern with two logarithmic spirals that look like the following:



In this representation, the seeds are centered on the intersections. It is not obvious that these intersections are, indeed, separated by the golden angle. Remember, two intersections that are supposed to be separated by 137.5° are in the order of their distance from the center. The intersections labeled A and B are in order from the center and they are separated by 137.5°.

In the following, the angles are clearer because there are fewer spirals:



Each of the intersections has been numbered in order from the outside. As you can confirm with a protractor, each intersection is 137.5° from the next.

A possible biological explanation for this pattern is possible. Imagine that there are some germinal cells in the center of the growing bud that generate buds that eventually result in seeds. If germinal cells spin so that they create successive buds at 137.5 degrees and the buds each grow linearly, then their distance from the center would grow quadracally. This would result in this spiral.

The spirals in the illustrations have been constructed to use Fibonacci numbers for the number of spirals. (The first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89... Each is the sum of the two preceding.) The first illustration was constructed with 13 spirals in one direction and 21 in the other; the second illustration used 5 and 8. It is claimed that this is how nature works, and that using such numbers is particularly attractive.

Hoping to produce a pretty logo for the Concord Consortium, I made a SuperCard script that can generate these patterns. One can specify the number of spirals in each direction, the total number of intersections, and the inner and outer radius in screen pixels. Developing this script was an interesting mathematical challenge.

The key breakthrough came one night in a train on the Canadian high plains when I awoke at 4 AM and couldn't get back to sleep. The problem I had been having was that I found it difficult to work with the logarithmic spirals. How was I to compute the steepness of the various spirals so that they would result in intersections that happened at just the golden angles? Could this be done for any arbitrary number of spirals?

The answer that came in a flash was to transform the problem into rectangular coordinates chosen so that spirals mapped into straight lines. I could easily solve the problem in this space because straight lines are easy to work with. If I selected the transformation correctly, then every straight line would map into a logarithmic spiral and the lattice-like solutions with straight lines in rectangular space would give the desired sunflower spirals when transformed.

So, let's first decide on the transformation, then solve the problem in rectangular space, and finally transform it into the target display coordinates.

The target coordinates are polar coordinates for the screen: r and α, where r is the distance from the center and α is the angle measured around the center. Of course, the computer needs rectangular coordinates, so eventually we will use

$h = h_0 + r \sin \alpha$

$v = v_0 + r \cos \alpha$

where h and v are horizontal screen coordinates and h_0 and v_0 are the coordinates of the center of the screen.

We will need to make the sunflower design so that it is contained between two circles of radius r_0 and r_1. Thus, we will have $r_0 \le r \le r_1$ and $0 \le \alpha \le 2\pi$.

In polar coordinates, a spiral can be described parametrically using

$r = at^2$ and $\alpha = t + t_0$ where $0 \le t$ for one direction spiral and $0 \ge t$ for the other.

Here a and t_0 characterize the particular spiral; a is a measure of how steep it is and t_0 rotates it around its center.

I will solve the problem in (u,v) space where v^2 will be proportional to r whereas u and α will be proportional. In this space, straight lines will map into spirals in polar coordinates. I found it convenient to solve the problem in this space using the bounds $0 \le u \le 1$ and $0 \le v \le N$ where N is the total number of intersections.

The transformations that accomplish this are:

$r = r_0 + (r_1 - r_0)(v/N)^2$

$\alpha = 2\pi u$

Note that the α axis is periodic in 2π, so the u axis is periodic with period 1. We want intersections every $2\pi/\varphi$ along the φ axis, so we want them every $1/\varphi$ along the u axis.

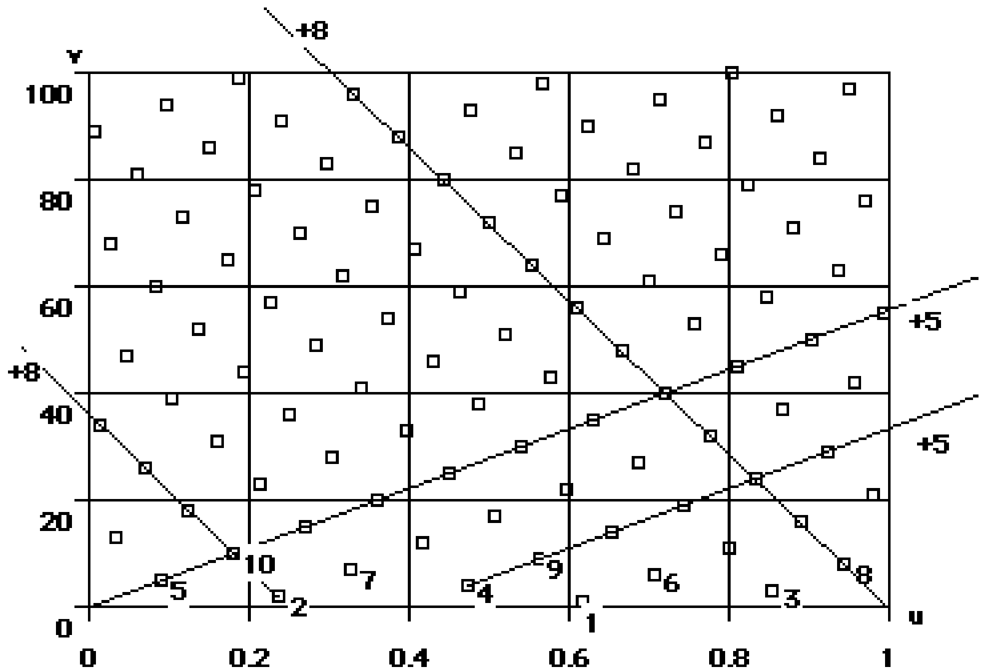
To see how the intersections would line up, I plotted the intersections that would be at points every 1/φ in the u direction in (u,v) space. Specifically, I plotted tiny squares at $u = i/\varphi \bmod 1$ and $v = i$ for the integers i = 1, 2, ... N.

The code that draws these is the following

```
on showg n
  grid 0,1,0,n
  choose line tool
  put 2/(1+sqrt(5)) into gi --the golden ratio inverse
  repeat with i=0 to n
    put (i*gi) mod 1 into x
    square x,i
  end repeat
end showg
```

The procedure `grid x0, x1, y0, y1` draws a graphing grid for the problem coordinates x and y where $x0 \le x \le x1$ and $y0 \le y \le y1$. The procedure `square x,y` draws a small square centered at problem coordinates x,y. The `gi` referred to in this code is 1/φ.

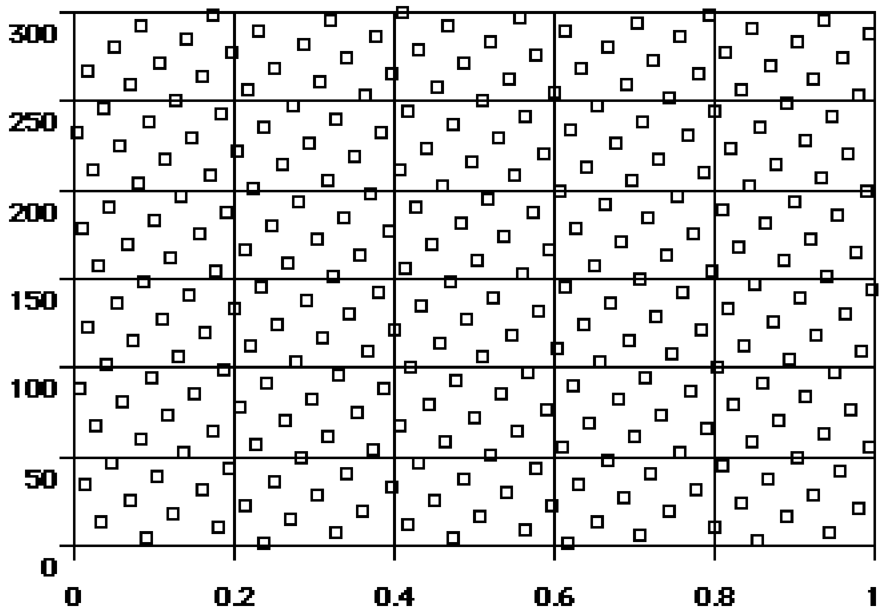
The resulting output for N = 100 (using `showg 100`) is shown below:



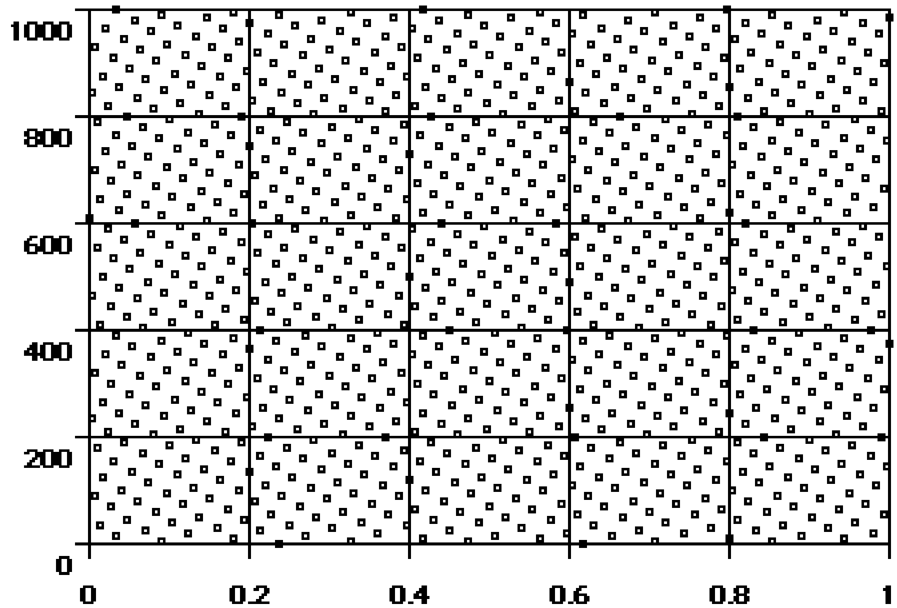
I have labeled the first ten intersections to emphasize how this grid is constructed and added some diagonal lines. Note how the intersections line up like tombstones in a cemetery; they line up in series of parallel straight lines. Since straight lines in this space map into spirals in screen space, these parallel lines result in the desired spirals.

There are an infinite number of sets of parallel lines one could draw through these intersections, but some are more **prominent** that others. On the grid above, two of the most prominent in each direction are shown. The ones slanting to the right connect points that are five apart and the ones slanting to the left connect points that are 8 apart. Not too surprisingly, there are five of the former and eight of the latter. What **is** surprising is that these are Fibonacci numbers. Is it always 5 and 8, or are other lines more prominent in other cases? If so, are they also Fibonacci numbers?

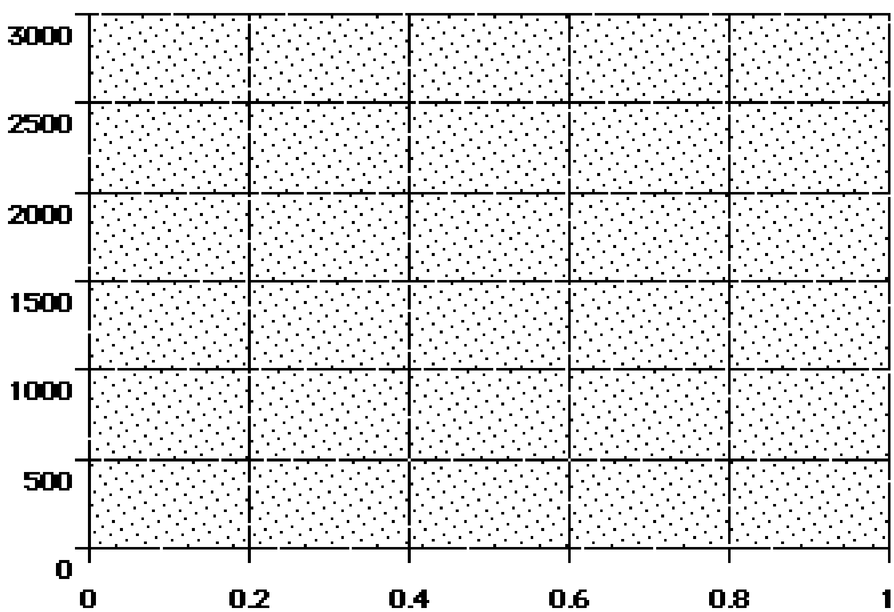
One important observation is that the lines that appear to be most prominent depend on the scale. Here is what 300 intersections look like:



Here lines corresponding to 13 and 21 are most prominent, and these are also Fibonacci numbers. The following is even more dense and now the 21 and 35 lines (still Fibonacci numbers) show up most clearly. (I made the squares smaller here)



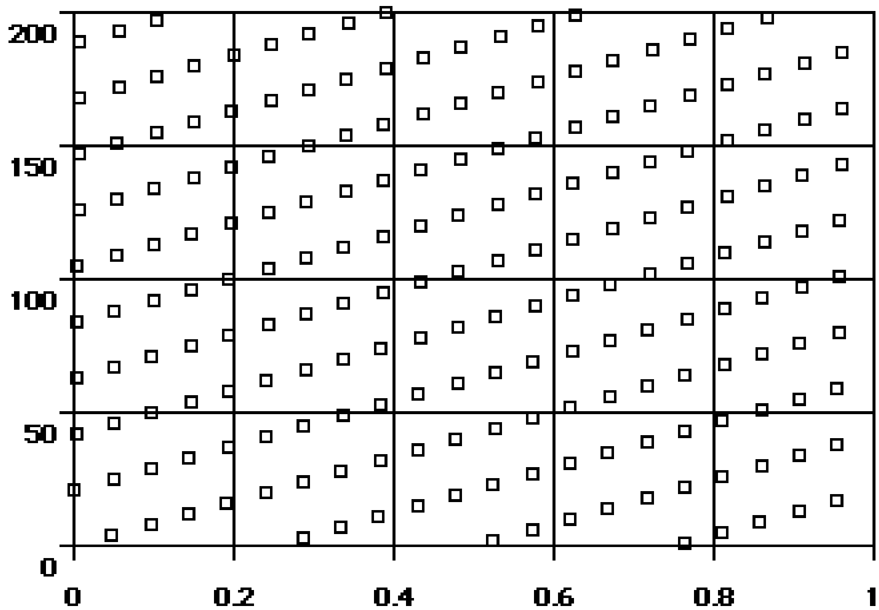
And, finally, the following is another where 35 and 55 are most prominent (here I have eliminated the squares completely).



What is happening in this series is that the patterns are getting squashed in the vertical direction; in the same graphing window, more and more points are being shown vertically. In spite of appearances, there is no change in the horizontal scale; the points seem more dense because they have been squashed vertically. As the points are increasingly squashed, what appear to be the most prominent lines change. The lines that appeared prominent for small N eventually become so nearly horizontal that our eye cannot even pick them out. On the other hand, lines that were almost vertical for low N become more nearly diagonal as N increases and therefore seem more prominent.

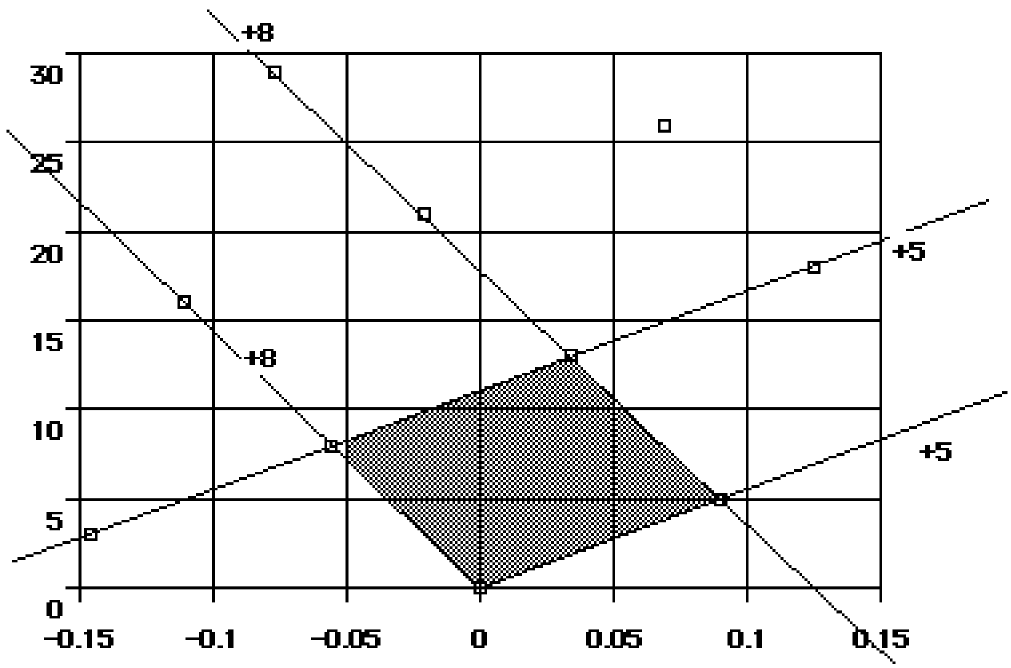
Will the most prominent lines always be Fibonacci numbers? Is there any connection between this appearance of the Fibonacci numbers and the golden ratio?

To answer this latter question, I constructed a similar pattern that did not use the golden ratio. The following uses π / √17 (an irrational I selected at random) in place of 1/φ.



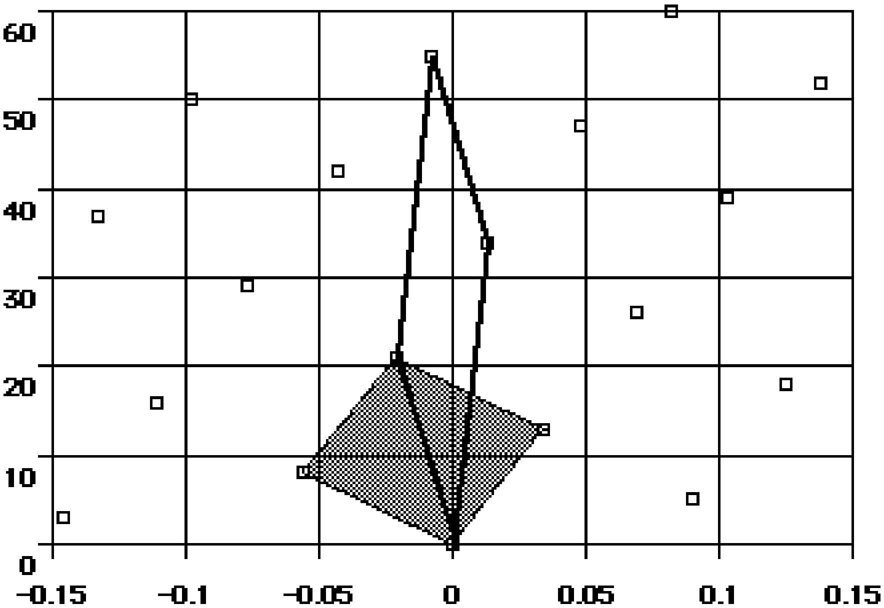
This pattern seems to have three prominent lines corresponding to +4 (the shallow lines rising to the right), +21 (the near-vertical lines), and +17, (the steep lines rising to the left). Of these, only the 21 is a Fibonacci number. Clearly, the appearance of Fibonacci numbers is related to the use of the golden ratio to construct the pattern.

To explore the meaning of "most prominent," I enlarged the section of these diagrams around the origin, and included a bit of the negative side, too. The illustration below shows a 3x magnification of the 100-point design that has +5 and +8 as the most prominent lines.

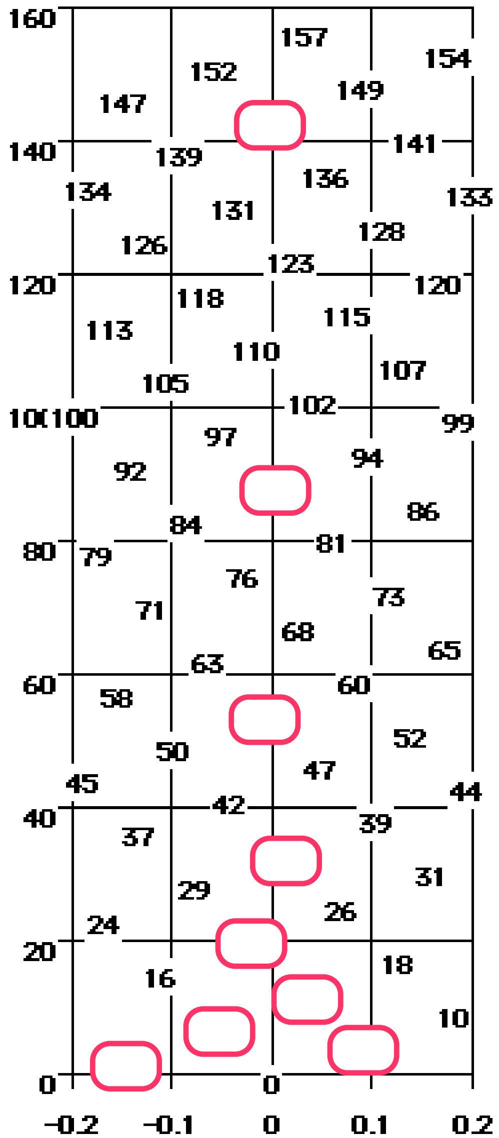


I have shaded a "central parallelogram", which I have defined as the parallelogram bounded by intersections that includes the origin and has the most prominent lines as its edges. There are other possible candidates for this central parallelogram, but they are all more acute, less square. I claim that **the most prominent lines are those that make this central parallelogram most square**. "Most prominent" is a perceptual issue and is subject to test. Let's assume that a series of tests would confirm my definition and see where that gets us.

Just as the most prominent lines will change as the vertical scale is changed, so does the central parallelogram. The illustration below shows an enlargement of the intersections for a 8-13 pattern with a shaded central parallelogram. One can imagine that, if the vertical scale was compressed, the parallelogram outlined in darker lines would become the most square one and the shaded one would be flattened beyond recognition. Interestingly, this new one is also constructed from Fibonacci lines, 21 and 34. (This can be read directly from the vertical heights of the left and right corners.)



This suggests that all the central parallelograms will be constructed from points near the v axis. It also suggests that all the points near the v axis are Fibonacci numbers.



To test this, I constructed a display that shows the region around u = 0 and plotted the number of the intersection instead of a square. The amazing result is that **all** the Fibonacci numbers line up near the v axis. In the accompanying illustration, I have circled all the Fibonacci numbers from 3 to 144. In each case they are the intersections nearest to the axis. The larger the number of the intersection, the closer it is to the axis. Since the foregoing suggests that the points nearest the axis become the sides of the central parallelograms, then it seems reasonable that it is the property of Fibonacci numbers that they are always nearest the axis that ensures that they always end up in the principle parallelograms.

So now the question becomes: Why are the numbers nearest the v-axis always Fibonacci numbers?

The answer to this question is related to the remarkable observation that the ratio of successive Fibonacci numbers approximates φ closely, getting very close as the terms get large.

Thus, to a very good approximation,

$F_n / F_{n+1} = \varphi$

or

$F_n / \varphi = F_{n+1}$

Here F_n is the n-th Fibonacci number and F_{n+1} is the next one.

Since we are plotting F_n / γ mod 1 on the u axis, and this is approximately the remainder of an integer (F_{n+1} mod 1), it will be close to zero. To get a feel for how close the fraction F_n / F_{n+1} will be to an integer, I divided each of the first 28 Fibonacci numbers by phi and took the absolute value of the difference between that and the preceding Fibonacci number. The result is shown in the following table.

i	Fi	Fi/phi	Distance
1	1	0.61803	
2	1	0.61803	0.381966
3	2	1.23607	0.236068
4	3	1.85410	0.145898
5	5	3.09017	0.090170
6	8	4.94427	0.055728
7	13	8.03444	0.034442
8	21	12.97871	0.021286
9	34	21.01316	0.013156
10	55	33.99187	0.008131
11	89	55.00502	0.005025
12	144	88.99689	0.003106
13	233	144.00192	0.001919
14	377	232.99881	0.001186
15	610	377.00073	0.000733
16	987	609.99955	0.000453
17	1597	987.00028	0.000280
18	2584	1596.99983	0.000173
19	4181	2584.00011	0.000107
20	6765	4180.99993	0.000066
21	10946	6765.00004	0.000041
22	17711	10945.99997	0.000025
23	28657	17711.00002	0.000016
24	46368	28656.99999	0.000010
25	75025	46368.00001	0.000006
26	121393	75025.00000	0.000004
27	196418	121393.00000	0.000002
28	317811	196418.00000	0.000001

It is clear from this table that the result of dividing a Fibonacci number by φ gets increasingly close to an integer for larger Fibonacci numbers.

This suggests that the horizontal positions of the Fibonacci numbers

$F_n/\varphi \bmod 1$

are close to the v axis but leaves open the question about whether they are the **closest**. To demonstrate this, I need to show that there are no nearby fractions that are closer.

For example, when n= 12, Fn = 144 and I would need to show that all other numbers k between F11 and F13 (that is 89 < k < 144) it is true that

$\mathrm{abs}(k/\varphi \bmod 1) < .003106$

I need help at this point. I know that this is a property of Fibonacci numbers, but I do not know the proof.